Generating functions for vector partition functions and a basic recurrence relation

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ABSTRACT
We define a generalized vector partition function and derive an identity for generating series of such functions associated with solutions of basic recurrence relation of combinatorial analysis. As a consequence we obtain the generating function of the number of generalized lattice paths and a new version of Chaundy-Bullard identity for the vector partition function.

KEYWORDS
Generating function; basic recurrence relation; vector partition function; Chaundy-Bullard identity

1. Statement of the main results

A basic identity in the theory of summation of functions is the following, practically tautological, relation:

$$\varphi(x) - \varphi(0) = \sum_{k=1}^{x} \left(\varphi(k) - \varphi(k-1)\right), x \in \mathbb{N}, \tag{1}$$

If for a given function $h(x)$ it is possible to find a function $\varphi(x)$, such that $\varphi(x) - \varphi(x-1) = h(x)$, then (1) becomes

$$\varphi(k)\Big|_{0}^{x} = \sum_{k=1}^{x} h(k), \tag{2}$$

which is the discrete analogue of the Newton-Leibniz formula, and the function $\varphi(x)$ is a discrete primitive for the function $h(x)$. For the generating function $\Phi(\xi) =$

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$\sum_{x=0}^{\infty} \varphi(x)\xi^x, \xi \in \mathbb{C}$, identity (1) is equivalent to

$$\Phi(\xi) - \frac{\Phi(0)}{1-\xi} = \frac{1}{1-\xi} \sum_{x=1}^{\infty} (\varphi(x) - \varphi(x-1))\, \xi^x \tag{3}$$

or

$$(1-\xi)\Phi(\xi) - \Phi(0) = \sum_{x=1}^{\infty} (\varphi(x) - \varphi(x-1))\, \xi^x. \tag{4}$$

In this paper we define a generalized vector partition function associated with a set of lattice vectors and with a complex-valued function of integer arguments. We give a multidimensional analogue of identity (4) in Theorem 1.1 and use it to investigate some properties of generalized lattice paths (Propositions 1.2 and 1.3) and prove a version of the Chaundy-Bullard identity for the generalized vector partition function (Proposition 1.4, see [3]).

Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ and $\Delta = \{\alpha^1, \alpha^2, \ldots, \alpha^N\} \subset \mathbb{Z}^n$ be a finite set of column vectors. Let $\mathbb{R}^n_{\geqslant}$ be a subset of $\mathbb{R}^n$ with non-negative coordinates. We let K denote the cone K spanned by the vectors in Δ:

$$K = \{\lambda \in \mathbb{R}^n : \lambda = x_1\alpha^1 + \cdots + x_N\alpha^N, x \in \mathbb{R}^N_{\geqslant}\}.$$

We assume that cone K is *pointed*, which means it does not contain any line or equivalently lies in an open half-space of $\mathbb{R}^n$. We let $A = [\alpha^1, \ldots, \alpha^N]$ denote $(n \times N)$-matrix composed of the column vectors in Δ.

The vector partition function $P_A(\lambda)$ of $\lambda \in \mathbb{Z}^n$ is (see, for example, [15]) the number of non-negative integer solutions to a linear Diophantine equation $\alpha^1 x_1 + \ldots + \alpha^N x_N = \lambda$:

$$P_A(\lambda) = \sum_{\substack{x:Ax=\lambda \\ x\in\mathbb{Z}^N_{\geqslant}}} 1, \quad \lambda \in \mathbb{Z}^n. \tag{5}$$

Geometrically the function $P_A(\lambda)$ equals the number of representations of the vector λ as a linear combination of the vectors in Δ with non-negative integer coefficients.

In [2] properties of the function

$$P_A(y; \lambda) = \sum_{\substack{x:Ax=\lambda \\ x\in\mathbb{Z}^N_{\geqslant}}} e^{-\langle x,y\rangle}, y \in \mathbb{C}^N, \tag{6}$$

called *the vector partition function associated with the set of vectors* Δ, are investigated. In particular, they derive the residue formulas for its generating function and an analog of the Euler-Maclaurin formula, in which the vector partition functions are represented as the action of the Todd operator on the volume function of a polyhedron.

Furthermore, a sum of $e^{\langle x,y\rangle}$ in integer cones was investigated in [14] in connection to generalization of the Riemann-Roch theorem. A structure theorem for vector partition function was presented and polyhedral tools for the efficient computation of such functions was provided in [16].

For an arbitrary function of integer arguments $\varphi : \mathbb{Z}^N_{\geqslant} \to \mathbb{C}$ we define a function

$$P_A(\lambda; \varphi) = \sum_{\substack{x: Ax = \lambda \\ x \in \mathbb{Z}^N_{\geqslant}}} \varphi(x), \lambda \in \mathbb{Z}^n$$

which we call *the vector partition function associated with* $\varphi(x)$.

For $\varphi(x) \equiv 1$, then the function $P_A(\lambda; \varphi)$ coincides with the classical function of the vector partition (5). For $\varphi(x) = e^{-\langle x,y \rangle}$ we obtain a vector partition function of the form (6). If we take $N = 2, A = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\varphi(x_1, x_2) = h(x_1)$, then $P_A(\lambda; \varphi) = \sum\limits_{\substack{x_1 + x_2 = \lambda \\ x_1, x_2 \geqslant 0}} h(x_1) = \sum\limits_{x_1 = 0}^{\lambda} h(x_1)$. Thus, the problem of finding the vector partition function $P_A(\lambda; \varphi)$ is a generalization of the classical summation's problem of functions of a discrete argument.

For further discussion and formulation of the main result we introduce some notations. Let $V = \{J\}$ be a set of all ordered sets $J = (j_1, j_2, \ldots, j_k)$, $1 \leqslant j_1 < \ldots < j_k \leqslant N, k = 0, 1, 2, \ldots, N$ and $\#J = k$ be a number of elements in the set J. Let π_j be the projection operator along the j-th coordinate axis in $\mathbb{R}^n$, i.e. $\pi_j x = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_N)$, and define its adjoint action on the function $\varphi(x) : \mathbb{Z}^N \to \mathbb{C}$ by: $\pi_j \varphi(x) = \varphi(\pi_j x), j = 1, \ldots, N$. Let $\mathbb{C}[[\xi]]$ be the ring of formal power series in the variable $\xi = (\xi_1, \ldots, \xi_N)$ and define the operator $\pi_j : \mathbb{C}[[\xi]] \to \mathbb{C}[[\xi]]$ for $j = 1, \ldots, N$ on the generating series $\Phi(\xi) = \sum\limits_{x \in \mathbb{Z}^N} \varphi(x) \xi^x$ as follows

$$\pi_j \Phi(\xi) = \sum_{x \in \mathbb{Z}^N} \varphi(\pi_j x) \xi^{\pi_j x} = \Phi(\xi_1, \ldots, \xi_{j-1}, 0, \xi_{j+1}, \ldots, \xi_N).$$

Furthermore, for $J \in V$ let $\pi_J = \pi_{j_1} \circ \cdots \circ \pi_{j_k}$ be a composition of operators $\pi_{j_1}, \ldots, \pi_{j_k}$ and $\pi_\emptyset = 1$ is the the identity operator.

For complex-valued functions $\varphi(x)$ of integer arguments $x = (x_1, \ldots, x_N)$ we define a shift operator δ_j for $j = 1, \ldots, N$ as follows

$$\delta_j \varphi(x) = \varphi(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_N)$$

and for $\mu = (\mu_1, \ldots, \mu_N) \in \mathbb{Z}^N$ we define $\delta^\mu = \delta_1^{\mu_1} \cdots \delta_N^{\mu_N}$.

For $c = (c_1, \ldots, c_N) \in \mathbb{C}^N$ we denote the operator $Q(\delta) = \delta_1 \cdot \ldots \cdot \delta_N - c_1 \delta_2 \cdot \ldots \cdot \delta_N - \cdots - c_N \delta_1 \cdot \ldots \cdot \delta_{N-1}$ and $z^A = (z^{\alpha^1}, \ldots, z^{\alpha^N})$.

Now we formulate a multidimensional analogue of identity (4).

Theorem 1.1. *Let $P_A(\lambda; \varphi)$ be the vector partition function, associated with a function $\varphi : \mathbb{Z}^N_{\geqslant} \to \mathbb{C}$, and $\Phi(\xi)$ be the generating series for $\varphi(x)$. Then*

$$\sum_{J \in V} (-1)^{\#J} \pi_J \left[(1 - \langle c, \xi \rangle) \Phi(\xi) \right] \Big|_{\xi = z^A} = \sum_{\lambda \in K \cap \mathbb{Z}^N} P_A(\lambda; Q(\delta) \varphi) z^\lambda. \qquad (7)$$

We note that for $N = 1$ and $c = 1$, formula (7) implies identity (4).

The problem of finding *the number of generalized lattice paths* is formulated as follows: find the number of paths on an integer lattice from the origin to the point $\lambda \in \mathbb{Z}^n$ using only steps in Δ.

A similar problem for $n = 2$ connected with the study of multidimensional difference equations and its application for generalized Dyck paths (see [10]) was considered in [1]. We note that the Cauchy problem for multidimensional difference equations was considered also in [12], [13] and some properties of generating function of its solution were investigated in [11].

A simple case of the problem above arises by choosing the set of steps which form an orthonormal basis in $\mathbb{R}^N$: $\alpha^j = e^j, j = 1, \ldots, N$, and the number $\varphi(x)$ of paths from the origin to the point $x \in \mathbb{Z}^N$ satisfies the basic recurrence relation of combinatorial analysis

$$(1 - \langle I, \delta^{-I}\rangle)\varphi(x) \equiv \varphi(x) - \varphi(x - e^1) - \cdots - \varphi(x - e^N) = 0 \tag{8}$$

where $x \in I + \mathbb{Z}^N_{\geqslant}$. For any solution of (8) we have:

Proposition 1.2. *If the function $\varphi(x)$ satisfies equation (8), then the associated vector partition function $P_A(\lambda; \varphi)$ satisfies the difference equation*

$$(1 - \langle I, \delta_\lambda^{-A}\rangle)P_A(\lambda; \varphi) \equiv P_A(\lambda; \varphi) - \sum_{j=1}^{N} P_A(\lambda - \alpha^j; \varphi) = 0.$$

The following proposition relates the number of lattice paths $\varphi(x)$ and the number of generalized lattice paths $P_A(\lambda; \varphi)$:

Proposition 1.3. *If $\varphi(x)$ is the number of lattice paths, then the associated the vector partition function $P_A(\lambda; \varphi)$ coincides with the number of generalized lattice paths with steps in Δ; in this case its generating function $F(z) = \sum\limits_{\lambda \in K \cap \mathbb{Z}^n} P_A(\lambda; \varphi) z^\lambda$ has the form*

$$F(z) = \frac{1}{1 - z^{\alpha^1} - z^{\alpha^2} - \cdots - z^{\alpha^N}}.$$

In 1960 T. Chaundy and J. Bullard in [3] considered the identity which is valid for $c_1, c_2 \in \mathbb{C}$ such that $c_1 + c_2 = 1$ and nonnegative integers μ_1 and μ_2

$$c_2^{\mu_2+1} \sum_{\nu_1=0}^{\mu_1} \binom{\mu_1 + \mu_2 - \nu_1}{\mu_1 - \nu_1} c_1^{\mu_1-\nu_1} + c_1^{\mu_1+1} \sum_{\nu_2=0}^{\mu_2} \binom{\mu_1 + \mu_2 - \nu_2}{\mu_2 - \nu_2} c_2^{\mu_2-\nu_2} \equiv 1.$$

This identity was subsequently found in approximation theory, nonrecursive digital filters [6], in the theory of wavelets [4], in the theory of Gauss hypergeometric functions. A detailed review, including various proofs of a one-dimensional case and a multidimensional analogues of this identity was given in [7], [8] and [5]. In [9] similar identities were derived by using methods of generating functions and properties of Hadamars composition of multiple power series.

We give an analogue of the Chaundy-Bullard identity for vector partition function. For $j = 1, \ldots, N$ we denote $\Delta_j = \Delta \setminus \{\alpha^j\}$ and $A_j = [\alpha^1, \ldots, \alpha^{j-1}, \alpha^{j+1}, \ldots, \alpha^N]$, then $K_j = \{\nu \in \mathbb{Z}^n : \nu = y_1\alpha^1 + \ldots [j] \ldots + y_N\alpha^N, y \in \mathbb{Z}^N_{\geqslant}\}$; $c^x = c_1^{x_1} \cdots c_N^{x_N}$. Note that each cone $K_j \subset K$ is also pointed.

Proposition 1.4. *If $c_1 + c_2 + \cdots + c_N = 1$ and $\varphi_j(x) = \frac{|x|!}{x!} c^{x+e^j}$, then for any*

$\mu = (\mu_1, \ldots, \mu_N) \in \mathbb{Z}^N$ *the identity*

$$\sum_{j=1}^{N} \sum_{\nu \in K_j} P_{A_j}(\nu) P_A(\mu - \nu; \varphi_j) = P_A(\mu) \tag{9}$$

takes place.

The sum on the left side of identity (9) is finite since all the cones $K_j, j = 1, \ldots, N$ are pointed.

Note that for $\alpha^j = e^j, j = 1, \ldots, N$ we obtain from (9) a multidimensional Chaundy-Bullard identity:

$$\sum_{j=1}^{N} \sum_{\substack{0 \leqslant \nu \leqslant \mu \\ \nu_j = 0}} \frac{(|\mu| - |\nu|)!}{(\mu - \nu)!} c^{\,\mu - \nu + e^j} \equiv 1,$$

where the double inequality $0 \leqslant \nu \leqslant \mu$ means that $0 \leqslant \nu_j \leqslant \mu_j$ for all $j = 1, \ldots, N$.

2. Proofs

In this section we prove the main result (see Theorem 1.1) and its corollaries. First, using Lemmas 2.1 and 2.2, we prove the identity (7) for a set of standard basis vectors, and then, using a monomial substitution, we prove Theorem 1.1. The proofs of Propositions 1.2 and 1.3 follow directly from the main result. Then, using Lemmas 2.3 and 2.4, we prove Proposition 1.4.

We prove identity (7) for the case when the set of vectors $\Delta = \{\alpha^1, \ldots, \alpha^N\}$ consists of unit vectors $\alpha^j = e^j$, where the vector $e^j = (0, \ldots, 0, 1, 0, \ldots, 0)$ contains a unit on the j-th place for $j = 1, \ldots, N$. Then the vector partition function $P_A(\lambda; \varphi) = \varphi(x)$, and the generating series $\Phi(\xi) = \sum\limits_{x \in \mathbb{Z}^N_{\geqslant}} \varphi(x) \xi^x$ in identity (7) takes the form

$$\sum_{J \in V} (-1)^{\# J} \pi_J \left[(1 - \langle c, \xi \rangle) \Phi(\xi)\right] = \sum_{x \in I + \mathbb{Z}^N_{\geqslant}} (1 - \langle c, \delta^{-I} \rangle) \varphi(x) \xi^x, \tag{10}$$

where $\langle c, \xi \rangle = c_1 \xi_1 + \cdots + c_N \xi_N$, and $I = (1, \ldots, 1)$.

To prove (10) we use following properties of the operator $\mathbf{\Pi} = \sum\limits_{J \in V} (-1)^{\# J} \pi_J$:

Lemma 2.1. *For an arbitrary series* $\Phi(\xi) = \sum\limits_{x \in \mathbb{Z}^N_{\geqslant}} \varphi(x) \xi^x$ *the operator* $\mathbf{\Pi}$ *acts on* $\Phi(\xi)$ *as follows*

$$\mathbf{\Pi} : \sum_{x \in \mathbb{Z}^N_{\geqslant}} \varphi(x) \xi^x \mapsto \sum_{x \in I + \mathbb{Z}^N_{\geqslant}} \varphi(x) \xi^x.$$

Proof. We represent the operator $\mathbf{\Pi}$ as a composition $\mathbf{\Pi} = (1 - \pi_1)(1 - \pi_2) \cdots (1 - \pi_N)$, where $1 = \pi_\emptyset$ is an identity operator, and use the commutativity of its factors to apply it to the series $\Phi(\xi)$:

$$\begin{aligned}
&(1-\pi_1)(1-\pi_2)\cdots(1-\pi_N)\Phi(\xi) = \\
&\qquad = (1-\pi_1)(1-\pi_2)\cdots(1-\pi_{N-1})\left[\Phi(\xi)-\Phi(\pi_N\xi)\right] = \\
&\qquad = (1-\pi_1)(1-\pi_2)\cdots(1-\pi_{N-1}) \sum_{x\in e^N+\mathbb{Z}^N_{\geqslant}} \varphi(x)\xi^x = \\
&\qquad = (1-\pi_1)(1-\pi_2)\cdots(1-\pi_{N-2}) \sum_{x\in e^N+e^{N-1}+\mathbb{Z}^N_{\geqslant}} \varphi(x)\xi^x = \\
&\qquad = \ldots = \sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \varphi(x)\xi^x,
\end{aligned}$$

where $I = e^1 + e^2 + \ldots + e^N = (1, 1, \ldots, 1)$. □

Lemma 2.2. *If $\mathbf{\Pi}_j = (1-\pi_1)\cdots(1-\pi_{j-1})(1-\pi_{j+1})\cdots(1-\pi_N)$, then for any $j = 1, \ldots, N$ the following equality holds*

$$\mathbf{\Pi}\xi_j\Phi(\xi) = \mathbf{\Pi}_j\xi_j\Phi(\xi) = \sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \varphi(x-e^j)\xi^x.$$

Proof. Similarly as in the proof of Lemma 2.1 we represent the operator $\mathbf{\Pi}$ as a composition and apply it to $\xi_j\Phi(\xi) \in \mathbb{C}[[\xi]]$:

$$\begin{aligned}
&(1-\pi_1)(1-\pi_2)\cdots(1-\pi_N)\left[\xi_j\Phi(\xi)\right] = \\
&\qquad = (1-\pi_1)\cdots(1-\pi_{j-1})(1-\pi_{j+1})\cdots(1-\pi_N)\left[(1-\pi_j)\xi_j\Phi(\xi)\right] = \\
&\qquad = (1-\pi_1)\cdots(1-\pi_{j-1})(1-\pi_{j+1})\cdots(1-\pi_N)\left[\xi_j\Phi(\xi)-\pi_j\xi_j\Phi(\xi)\right] = \\
&\qquad = (1-\pi_1)\cdots(1-\pi_{j-1})(1-\pi_{j+1})\cdots(1-\pi_N)\left[\xi_j\Phi(\xi)\right] = \\
&\qquad = \mathbf{\Pi}_j \sum_{x\in e^j+\mathbb{Z}^N_{\geqslant}} \varphi(x-e^j)\xi^x = \sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \varphi(x-e^j)\xi^x.
\end{aligned}$$

□

By using Lemma 2.1 and 2.2, we now prove (10).

Proof. We apply the operator $\mathbf{\Pi}$ to the product $(1-\langle c,\xi\rangle)\Phi(\xi)$ and use Lemmas 2.1 and 2.2 to obtain:

$$\begin{aligned}
&\mathbf{\Pi}\left[(1-\langle c,\xi\rangle)\Phi(\xi)\right] = \mathbf{\Pi}\Phi(\xi) - \mathbf{\Pi}\left[\langle c,\xi\rangle\Phi(\xi)\right] = \mathbf{\Pi}\Phi(\xi) - \langle c,\mathbf{\Pi}\xi\rangle\Phi(\xi) = \\
&\qquad = \mathbf{\Pi}\Phi(\xi) - c_1\mathbf{\Pi}_1\xi_1\Phi(\xi) - \cdots - c_N\mathbf{\Pi}_N\xi_N\Phi(\xi) = \\
&\qquad = \sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \varphi(x)\xi^x - c_1\sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \varphi(x-e^1)\xi^x - \ldots - c_N\sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \varphi(x-e^N)\xi^x = \\
&\qquad = \sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \left[\varphi(x) - c_1\varphi(x-e^1) - \ldots - c_N\varphi(x-e^N)\right]\xi^x = \\
&\qquad = \sum_{x\in I+\mathbb{Z}^N_{\geqslant}} \left[(1-\langle c,\delta^{-I}\rangle)\varphi(x)\right]\xi^x.
\end{aligned}$$

$\square$

Now we proceed to the proof of the main theorem.

Proof. We substitute the monomial replacement of the variables $\xi = z^A$ in the formula (10) to obtain

$$\xi^x = (z_1^{\alpha_1^1}\cdots z_n^{\alpha_n^1})^{x_1}\ldots(z_1^{\alpha_1^N}\cdots z_n^{\alpha_n^N})^{x_N} = z_1^{x_1\alpha_1^1+\ldots+x_N\alpha_1^N}\cdots z_n^{x_1\alpha_n^1+\ldots+x_N\alpha_n^N} = z^\lambda,$$

where $\lambda = Ax$. We further observe that if $x \in I + \mathbb{Z}^N_{\geqslant}$, then $\lambda \in K \cap \mathbb{Z}^n$. Therefore,

$$\sum_{J\in V}(-1)^{\#J}\pi_J\left[(1-\langle c,\xi\rangle)\Phi(\xi)\right]\Big|_{\xi=z^A} =$$

$$= \sum_{x\in I+\mathbb{Z}^N_{\geqslant}}\left[\varphi(x)-c_1\varphi(x-e^1)-\ldots-c_N\varphi(x-e^N)\right]\xi^x\Big|_{\xi=z^A} =$$

$$= \sum_{\lambda\in K\cap\mathbb{Z}^n}\sum_{\substack{Ax=\lambda\\ x\in\mathbb{Z}^N_{\geqslant}}}\left[\varphi(x+I)-c_1\varphi(x+I-e^1)-\ldots-c_N\varphi(x+I-e^N)\right]z^\lambda =$$

$$= \sum_{\lambda\in K\cap\mathbb{Z}^n}P_A\left(\lambda;Q(\delta)\varphi\right)z^\lambda,$$

where $Q(\delta) = \delta_1\cdot\ldots\cdot\delta_N - c_1\delta_2\cdot\ldots\cdot\delta_N - \cdots - c_N\delta_1\cdot\ldots\cdot\delta_{N-1}$. $\square$

Now we prove Proposition 1.2.

Proof. Summing the left side of the basic recurrence relation (8) over all integer nonnegative $x : Ax = \lambda$, we obtain

$$\sum_{x:Ax=\lambda}\varphi(x) - \sum_{x:Ax=\lambda}\varphi(x-e^1) - \cdots - \sum_{x:Ax=\lambda}\varphi(x-e^N) = 0. \tag{11}$$

Note that for any $j = 1,\ldots,N$ we have

$$\sum_{\substack{x:Ax=\lambda\\ x\geqslant 0}}\varphi(x-e^j) = \sum_{\substack{x:A(x-e^j+e^j)=\lambda\\ x\geqslant 0}}\varphi(x-e^j) = \sum_{\substack{x:A(x-e^j)=\lambda-\alpha^j\\ x\geqslant 0}}\varphi(x-e^j) =$$

$$= \sum_{\substack{x:Ax=\lambda-\alpha^j\\ x\geqslant -e^j}}\varphi(x) = \sum_{\substack{x:Ax=\lambda-\alpha^j\\ x\geqslant 0}}\varphi(x) - \sum_{\substack{x:Ax=\lambda-\alpha^j\\ x_j=-e^j}}\varphi(x) =$$

$$= P_A(\lambda-\alpha^j;\varphi),$$

since $\varphi(x) = 0$ for all $x \notin \mathbb{Z}^N_{\geqslant}$. Then from (11) we obtain

$$P_A(\lambda;\varphi) - P_A(\lambda-\alpha^1;\varphi) - \ldots - P_A(\lambda-\alpha^N;\varphi) = 0.$$

$\square$

Now we prove Preposition 1.3.

Proof. We prove that the generating function $\Phi(\xi)$ of the number of lattice paths is equal to $\Phi(\xi) = (1 - \xi_1 - \xi_2 - \ldots - \xi_N)^{-1}$. Indeed, the number $\varphi(x)$ of lattice paths satisfies the basic recurrence relation (8). Therefore, the right side of (10) vanishes. Now we use induction on the number N of variables $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$. For $N = 1$ formula (10) takes the form $(1-\xi)\Phi(\xi) - 1 = 0$, where $\Phi(\xi) = (1-\xi)^{-1}$ and for any number $m < N$ of variables $(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_m})$ the generating function $\Phi(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_m}) = (1 - \xi_{i_1} - \xi_{i_2} - \cdots - \xi_{i_m})^{-1}$.

We note that for any $J = \{j_1, j_2, \ldots, j_k\}, k \leqslant N, J \neq \emptyset$ the number of lattice paths in $\mathbb{Z}^{N-k} = \mathbb{Z}^N \cap \{x_{j_1} = \cdots = x_{j_k} = 0\}$ can be written as a function $\pi_J \varphi(x)$, and the induction hypothesis implies that its generating function $\Phi(\xi)$ satisfies the relation $\pi_J \Phi(\xi) = \pi_J (1 - \xi_1 - \cdots - \xi_N)^{-1}$ or $\pi_J \left[(1 - \xi_1 - \cdots - \xi_N)\Phi(\xi)\right] = 1$.

Next, we select in (10) the term corresponding to $J = \emptyset$:

$$(1 - \xi_1 - \ldots - \xi_N)\Phi(\xi) + \sum_{\substack{J \neq \emptyset \\ J \in V}} (-1)^{\# J} 1 = 0.$$

The equality $\sum\limits_{\substack{J \neq \emptyset \\ J \in V}} (-1)^{\# J} 1 = -C^N_{N-1} + C^N_{N-2} + \cdots + (-1)^N C^N_0 = -1$ implies that the induction statement is also true for $m = N$. After making the substitution $\xi = z^A$ we obtain Proposition 1.3. $\square$

The following two lemmas are required to prove the Proposition 1.4.

Lemma 2.3. *If the function $\Phi(\xi)$ does not depend on the variable ξ_j, then $\mathbf{\Pi}\Phi(\xi) = 0$.*

Proof. We consider the operator $\mathbf{\Pi}_j = (1 - \pi_1) \circ \cdots [j] \cdots \circ (1 - \pi_N), j = 1, \ldots, N$. Since the function $\Phi(\xi)$ does not depend on the variable ξ_j, we have $(1 - \pi_j)\Phi(\xi) = \Phi(\xi) - \pi_j \Phi(\xi) = 0$, therefore $\mathbf{\Pi}\Phi(\xi) = \mathbf{\Pi}_j (1 - \pi_j)\Phi(\xi) = 0$. $\square$

Lemma 2.4. *For any complex values $c_1, \ldots, c_N$ such that $c_1 + \cdots + c_N = 1$, the rational fraction $(I - \xi)^{-1}$ can be represented as follows:*

$$\frac{1}{I - \xi} = \sum_{j=1}^{N} \frac{c_j}{\pi_j (I - \xi)} \cdot \frac{1}{1 - \langle c, \xi \rangle}. \tag{12}$$

Proof. Since $c_1 + \cdots + c_N = 1$ and $\varphi(x) \equiv 1$, the right side of (10) vanishes, and its left side can be written as follows:

$$(1 - \langle c, \xi \rangle) \frac{1}{I - \xi} + \sum_{J \neq \emptyset} (-1)^{\# J} \pi_J (1 - \langle c, \xi \rangle)\Phi(\xi) = 0.$$

Since $\sum\limits_{J \neq \emptyset} (-1)^{\# J} \pi_J = \mathbf{\Pi} - 1$ and $(1 - \langle c, \xi \rangle) = \sum\limits_{j=1}^{N} c_j (1 - \xi_j)$, we obtain

$$\frac{1 - \langle c, \xi \rangle}{I - \xi} + \sum_{j=1}^{N} \mathbf{\Pi} \frac{c_j}{\pi_j (I - \xi)} = \sum_{j=1}^{N} \frac{c_j}{\pi_j (I - \xi)}.$$

Furthermore, Lemma 2.3 implies

$$\mathbf{\Pi}\left(\pi_j \frac{c_j}{I-\xi}\right) = 0, j = 1, \ldots, N,$$

which completed the proof. $\square$

Now we prove the Chaundy-Bullard identity for the vector partition function (Proposition 1.4).

Proof. We expand both sides of (12) in power series in ξ^μ to obtain

$$\sum_{\mu \in \mathbb{Z}^N_{\geqslant}} \xi^\mu = \sum_{j=1}^{N} \frac{c_j}{\pi_j(I-\xi)} \sum_{\mu \in \mathbb{Z}^N_{\geqslant}} \frac{c^\mu |\mu|!}{\mu!} \xi^\mu. \tag{13}$$

We recall that $K = \{\mu \in \mathbb{Z}^n : \mu = x_1\alpha^1 + \ldots + x_N\alpha^N, x \in \mathbb{Z}^N_{\geqslant}\}, K_j = \{\nu \in \mathbb{Z}^n : \nu = y_1\alpha^1 + \ldots [j] \ldots + y_N\alpha^N, y \in \mathbb{Z}^N_{\geqslant}\}$, and $K_j \subset K$ for $j = 1, \ldots, N$. In (13) we substitute $\xi = z^A$ and transform the left side as follows

$$\frac{1}{I - z^A} = \sum_{x \in \mathbb{Z}^n_{\geqslant}} z^{Ax} = \sum_{\mu \in K} \left(\sum_{\substack{x: Ax = \mu \\ x \in \mathbb{Z}^n_{\geqslant}}} 1 \right) z^\lambda = \sum_{\mu \in K} P_A(\mu) z^\mu.$$

We denote $\varphi_j(x) = \frac{|x|!}{x!} c^{x+e^j}$, then the right side of (13) takes the form

$$\sum_{j=1}^{N} c_j \pi_j (I-\xi)^{-1} (1 - \langle c, \xi \rangle)^{-1} = \sum_{j=1}^{N} \left(\sum_{\substack{y \geqslant 0 \\ y_j = 0}} \xi^y \cdot \sum_{x \in \mathbb{Z}^N_{\geqslant}} \varphi_j(x) \right) =$$

$$= \sum_{j=1}^{N} \left(\sum_{\nu \in K_j} \left(\sum_{\substack{y: Ay = \nu \\ y_j = 0}} 1 \right) z^\nu \cdot \sum_{\lambda \in K} \left(\sum_{x: Ax = \lambda} \varphi_j(x) \right) z^\lambda \right) =$$

$$= \sum_{j=1}^{N} \left(\sum_{\nu \in K_j} P_{\Delta_j}(\nu) z^\nu \cdot \sum_{\lambda \in K} P_\Delta(\lambda; \varphi_j(x)) z^\lambda \right) =$$

$$= \sum_{\mu \in K} \left(\sum_{j=1}^{N} \sum_{\substack{\nu + \lambda = \mu \\ \nu \in K_j \\ \lambda \in K}} P_{\Delta_j}(\nu) P_\Delta(\lambda; \varphi_j(x)) \right) z^\mu.$$

Equating the coefficients of z^μ yields

$$P_A(\mu) = \sum_{j=1}^{N} \sum_{\substack{\nu+\lambda=\mu \\ \nu\in K_j \\ \lambda\in K}} P_{A_j}(\nu) P_A(\lambda; \varphi_j(x)) = \sum_{j=1}^{N} \sum_{\nu\in K_j} P_{A_j}(\nu) P_A(\mu-\nu; \varphi_j(x)),$$

which completes the proof. $\square$

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